

# Quantitative Wave-Particle Duality

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The complementary wave and particle character of quantum objects (or quantons) was pointed out by Niels Bohr. This wave-particle duality, in the context of the two-slit experiment, is now described not just as two extreme cases of wave and particle characteristics, but in terms of quantitative measures of these natures. These measures of wave and particle aspects are known to follow a duality relation. A very simple and intuitive derivation of a closely related duality relation is presented, which should be understandable to the introductory student.

## I. INTRODUCTION

The two slit experiment with particles is perhaps one of the most beautiful experiments in physics. Particles going through a double-slit one by one, accumulate to form an interference pattern.<sup>1</sup> With advancement of technology, multislit diffraction has been demonstrated with large molecules like  $C_{60}$ .<sup>2</sup> Any quantum entity that shows the properties of both a particle and a wave, is often called a quanton,<sup>3</sup> and we will use this nomenclature in this paper. Although the quantons go through the double-slit one at a time, the accumulated result of many quantons shows an interference. It indicates that each quanton behaves like a wave and interferes with itself.

Objects like electrons are supposed to be “indivisible” particles, unlike photons and phonons which one believes to be actually waves, but occasionally localized into packets which look like particles. Being “indivisible” particles, one would naively imagine that these quantons pass through one of the two slits, and not both. However, if they yield an interference pattern, these particles must somehow be interacting with both the slits at the same time, just like a wave. Trying to resolve this issue, if one tries to experimentally find out which of the two slits the quanton passed through, the interference disappears. The quanton passing through one of the two slits, is associated with the particle nature, whereas the interference seen in cumulative results is associated with the wave nature. Wave and particle natures can only be seen one at a time, never simultaneously. This was formalized by Niels Bohr as the principle of complementarity.<sup>4</sup> Einstein had tried to argue against such a principle, and had proposed a thought experiment which, he claimed, showed wave and particle natures in the same experiment.<sup>5</sup> Bohr pointed out the flaw in Einstein’s argument and the principle of complementarity stood its ground.

It was recognized later that the wave and particles natures are not mutually exclusive in the strict sense of the word. It is possible to get partial information on which slit a quanton went through, and get an interference pattern which is not sharp. In other words wave-particle duality in Bohr’s principle can be stated, not just in terms of mutual exclusivity of purely particle and purely wave natures, but in terms of quantitative measures of these natures.<sup>6–9</sup> Wave-particle duality in terms of these quan-

titative measures of wave and particle natures is now described by the following duality relation<sup>9</sup>

$$\mathcal{V}^2 + \mathcal{D}^2 \leq 1, \quad (1)$$

where  $\mathcal{D}$  is a path distinguishability and  $\mathcal{V}$  the visibility of the interference pattern. Both these quantities vary between 0 and 1, and the above relation quantifies how visible will the interference be if the two paths through the two slits can be distinguished with a distinguishability  $\mathcal{D}$ . It should be emphasized here that distinguishability is necessarily associated with a measuring apparatus used to find out which of the two slits a quanton went through. Distinguishability  $\mathcal{D}$  being 1 would imply that one can say with absolute certainty which of the two slits the quanton went through. The quanton going through a particular slit implies that it behaves like a particle, in the sense described in the preceding paragraph. No interference is seen in such a scenario, as the interference visibility  $\mathcal{V}$ , as seen from (1), can only be 0. On the other hand, if  $\mathcal{V}$  is 1, the path distinguishability  $\mathcal{D}$  can only be 0. In this situation one cannot tell which of the two slits the quanton went through. The reader might wonder if this is just the inability of the experimenter to tell which slit the quanton went through, and if the quanton actually takes one path or the other. We would like to assert here that it is incorrect to assume that the quanton always takes one path or the other. Some prefer to assume that, left to itself, the quanton will take both the paths, although quantum mechanics does not say anything about it. Strictly speaking, one should assume the following. Even if we have made a measurement on a hundred quantons, each of which has told us which slit each of them passed through, we cannot say if the next quanton went through only one of the two slits (irrespective of which) without making a measurement. Unperformed experiments have no results.<sup>10</sup>

Going further, one might wonder what partial distinguishability ( $\mathcal{D}$  having value between 0 and 1) means for a single quanton, and whether each quanton contributes partially to interference. Or does the duality relation mean that some quantons give full which-path information and do not contribute to interference and some give zero which-path information and contribute fully to interference? These are the questions a student is likely to wonder, since the interference is built up by quantons going through one by one. We would like to address this

question and suggest a picture to visualize. In the following, we analyze a thought modification of the two-slit experiment with provision for path detection, and give a novel derivation of a duality relation closely related to Eq. (1).

## II. FULL WHICH-PATH INFORMATION

It can be demonstrated quite simply that if there is a path-detector which gains information about which of the two slits the quanton passed through, the interference will be completely destroyed.<sup>11,12</sup> Suppose that the state of a quanton, passing through a double-slit, is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle), \quad (2)$$

where  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are the amplitudes of the quanton passing through slit 1 and 2, respectively. The probability of finding the quanton at a point  $x$  on the screen, is given by

$$|\langle x|\Psi\rangle|^2 = \frac{1}{2}(|\langle x|\psi_1\rangle|^2 + |\langle x|\psi_2\rangle|^2 + \langle\psi_1|x\rangle\langle x|\psi_2\rangle + \langle\psi_2|x\rangle\langle x|\psi_1\rangle). \quad (3)$$

In the above,  $\langle x|\Psi\rangle$  is the wavefunction, also represented by  $\Psi(x)$ . The last two terms represent interference. Now, let us suppose that we have a quantum path-detector also included in the setup. Without going into the details of what such a path-detector might look like, we just assume that a quanton going through slit 1 leaves the detector in a state  $|d_1\rangle$  and that going through slit 2 leaves the detector in a state  $|d_2\rangle$ . If this detector is capable of detecting which path the quanton went through, according to von Neumann's recipe of a quantum measurement, the states of the detector should get correlated with the two paths of the quanton.<sup>13</sup> The state of the quanton and the path-detector combined, will necessarily be entangled, and will be of a form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle). \quad (4)$$

Measuring an observable which has two eigenstates  $|d_1\rangle, |d_2\rangle$ , with different eigenvalues, will indicate that the quanton went through a particular slit. A measurement of that observable yielding  $|d_1\rangle$  will lead to a definite conclusion that the quanton passed through slit 1, and likewise for  $|d_2\rangle$ . This, however, does not imply that the quanton goes through only one of the two slits, even when no path-detection is made. The probability of finding the quanton at a position  $x$ , is given by

$$|\langle x|\Psi\rangle|^2 = \frac{1}{2}(|\langle x|\psi_1\rangle|^2\langle d_1|d_1\rangle + |\langle x|\psi_2\rangle|^2\langle d_2|d_2\rangle + \langle\psi_1|x\rangle\langle x|\psi_2\rangle\langle d_1|d_2\rangle + \langle\psi_2|x\rangle\langle x|\psi_1\rangle\langle d_2|d_1\rangle). \quad (5)$$

The last two terms which would have given interference, are killed by the orthogonality of  $|d_1\rangle$  and  $|d_2\rangle$ , and one is reduced to

$$|\langle x|\Psi\rangle|^2 = \frac{1}{2}(|\langle x|\psi_1\rangle|^2 + |\langle x|\psi_2\rangle|^2). \quad (6)$$

An important point to be noted here is that the actual path measurement is not even necessary here. The mere existence of which-way information, or mere possibility of a path measurement, is enough to destroy interference.

The preceding discussion implies that finding the detector state (say)  $|d_1\rangle$  indicates that the quanton went through slit 1. But one may wonder if the quanton *really* went through slit 1, and if one can verify that. The problem becomes more complicated if the quanton hits the screen first and the path-detector states are looked at later. These are complex interpretational issues, and a consensus on these has not been arrived at. We would not delve into this issue here, and only refer the reader to a debate on it.<sup>14,15</sup> For our purpose, we would continue to assume that finding a path-detector state  $|d_1\rangle$  would imply quanton going through the upper slit, to the extent that is implied by the correlated state (4).

## III. PARTIAL WHICH-PATH INFORMATION

### A. Unambiguous quantum state discrimination

Suppose now that the path-detector has states  $|d_1\rangle$  and  $|d_2\rangle$  which are not orthogonal. In such a situation, no observable exists for which  $|d_1\rangle$  and  $|d_2\rangle$  are eigenstates with different eigenvalues. Hence it is not possible to distinguish between the two states perfectly. Consequently they cannot be used to distinguish between the two paths of the quanton. As example, recently a two-slit interference experiment was carried out at the molecular level where the atomic slits were movable. Depending on which slit the particle passed through, the atomic slits could recoil in two possible directions and thus carry the information about the passing particle.<sup>16</sup>

In this case the role of  $|d_1\rangle$  and  $|d_2\rangle$  is played by the momentum states of the counter-propagating atomic slits. If the two momentum states are not distinct, or orthogonal, they cannot be used to discern which slit the particle went through.

In the following we will describe a strategy which is the best one to distinguish between two non-orthogonal states *without error*. It goes by the name of unambiguous quantum state discrimination (UQSD).<sup>17-21</sup>

The state of the quanton plus the path-detector is still given by (4), but now  $|\langle d_1|d_2\rangle| \neq 0$ . For simplifying the analysis, we assume that  $\langle d_1|d_2\rangle$  is real and non-negative. Since the detector design is in the experimenter's control, this can always be arranged. Let there be a two-state *ancilla* system which interacts with the path-detector. This

interaction is characterized by the following properties,

$$\begin{aligned} \mathbf{U}_a|d_1\rangle|a_0\rangle &= \alpha|p_1\rangle|a_1\rangle + \beta|q\rangle|a_2\rangle \\ \text{and} \\ \mathbf{U}_a|d_2\rangle|a_0\rangle &= \alpha|p_2\rangle|a_1\rangle + \beta|q\rangle|a_2\rangle, \end{aligned} \quad (7)$$

where the path-detector states  $\langle p_1|p_2\rangle = 0$ , and the ancilla states  $\langle a_1|a_2\rangle = 0$ . It can be shown that as long as  $\langle d_1|d_2\rangle$  is real and non-negative, such an interaction always exists.<sup>17–19</sup> The two parameter can be easily shown to be  $|\beta|^2 = \langle d_1|d_2\rangle$  and  $|\alpha|^2 = 1 - \langle d_1|d_2\rangle$ .

Now, if in the measurement of the ancilla, one gets the state  $|a_1\rangle$ , the corresponding path-detector states will be  $|p_1\rangle$  or  $|p_2\rangle$ , which are orthogonal. Measuring a suitable observable of the path-detector will unambiguously tell us whether the state is  $|p_1\rangle$  or  $|p_2\rangle$ , and consequently, whether the original state was  $|d_1\rangle$  or  $|d_2\rangle$ . However, if the ancilla measurement yields  $|a_2\rangle$ , the corresponding path-detector states  $|q\rangle$  are identical, and one cannot distinguish between the two. In this case the process of distinguishing between  $|d_1\rangle$  and  $|d_2\rangle$  fails. If the states  $|d_1\rangle$  and  $|d_2\rangle$  occur with probability 1/2 each, the probability of failure to distinguish is just  $|\beta|^2 = \langle d_1|d_2\rangle$ .

Hence the probability of successfully *unambiguously* distinguishing between  $|d_1\rangle$  and  $|d_2\rangle$  is

$$P = 1 - \langle d_1|d_2\rangle. \quad (8)$$

The probability given by the above relation is the maximal probability with which an apparatus interacting with the d-system can *unambiguously* answer whether it is in state  $|d_1\rangle$  or  $|d_2\rangle$ .<sup>19</sup>

### B. Distinguishing between the quanton paths

Coming back to our problem of distinguishing between the two paths of the quanton, let us start from the state Eq. (4). If, for the time being, one is only interested in the state of the path-detector, one may ignore the states of the quanton. Thus, the reduced density matrix is

$$\begin{aligned} \rho_r &= \sum_{i=1}^2 \langle \psi_i | \Psi \rangle \langle \Psi | \psi_i \rangle \\ &= \frac{1}{2} |d_1\rangle \langle d_1| + \frac{1}{2} |d_2\rangle \langle d_2|, \end{aligned} \quad (9)$$

where we have used the orthogonality of  $|\psi_1\rangle$  and  $|\psi_2\rangle$ . The above indicates that the path-detector, for all practical purposes, is in a mixed state, where it may be thought to be randomly occurring be in state  $|d_1\rangle$  or  $|d_2\rangle$  with equal probability, provided one can distinguish between the two. The problem then reduces to distinguishing whether one has  $|d_1\rangle$  or  $|d_2\rangle$ , and UQSD is suited to provide an answer in this situation.

There is a subtle assumption in this interpretation that the states  $|d_1\rangle$  or  $|d_2\rangle$  occur as pure states randomly, whereas they actually occur in the entangled state

Eq. (4). This assumption may be justified by the observation that if one is dealing with a entangled system, by looking at just one part of the system, one cannot distinguish between a mixed state and an entangled state.

We let the ancilla interact with the path-detector, with the starting state being written as

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle)|a_0\rangle. \quad (10)$$

The interaction operator, for the ancilla and the path-detector, acts on the full entangled state, and the result is

$$\begin{aligned} |\Psi_f\rangle &= \mathbf{U}_a|\Psi_i\rangle \\ &= \frac{1}{\sqrt{2}} \mathbf{U}_a(|\psi_1\rangle|d_1\rangle + |\psi_2\rangle|d_2\rangle)|a_0\rangle \\ &= \frac{1}{\sqrt{2}} |\psi_1\rangle (\alpha|p_1\rangle|a_1\rangle + \beta|q\rangle|a_2\rangle) \\ &\quad + \frac{1}{\sqrt{2}} |\psi_2\rangle (\alpha|p_2\rangle|a_1\rangle + \beta|q\rangle|a_2\rangle) \\ &= \frac{\sqrt{1 - \langle d_1|d_2\rangle}}{\sqrt{2}} (|\psi_1\rangle|p_1\rangle + |\psi_2\rangle|p_2\rangle)|a_1\rangle \\ &\quad + \frac{\sqrt{\langle d_1|d_2\rangle}}{\sqrt{2}} (|\psi_1\rangle + |\psi_2\rangle)|q\rangle|a_2\rangle. \end{aligned} \quad (11)$$

First we make sure that introducing the ancilla system does not affect the visibility of interference. The probability density of the quanton hitting the screen at a position  $x$  is given by

$$\begin{aligned} |\langle x | \Psi_f \rangle|^2 &= (1 - \langle d_1|d_2\rangle) \frac{1}{2} (|\langle x | \psi_1 \rangle|^2 |\langle p_1 | p_1 \rangle|^2 \\ &\quad + |\langle x | \psi_2 \rangle|^2 |\langle p_2 | p_2 \rangle|^2) \\ &\quad + \langle d_1|d_2\rangle \frac{1}{2} (|\langle x | \psi_1 \rangle|^2 + |\langle x | \psi_2 \rangle|^2 \\ &\quad + \langle x | \psi_1 \rangle \langle \psi_2 | x \rangle + \langle x | \psi_2 \rangle \langle \psi_1 | x \rangle) |\langle q | q \rangle|^2 \\ &= \frac{1}{2} (|\langle x | \psi_1 \rangle|^2 + |\langle x | \psi_2 \rangle|^2 \\ &\quad + \langle d_1|d_2\rangle \{ \langle x | \psi_1 \rangle \langle \psi_2 | x \rangle + \langle x | \psi_2 \rangle \langle \psi_1 | x \rangle \}), \end{aligned} \quad (12)$$

where we have used the fact that  $|p_1\rangle, |p_2\rangle, |q\rangle$  are normalized. Visibility of the interference fringes is conventionally defined as  $\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$ , where  $I_{max}$  and  $I_{min}$  are the maximum and minimum values of intensity in a close neighborhood.<sup>22</sup> In a two-slit interference, the ideal visibility can be worked out to be just the factor multiplying the term  $\psi_1^*(x)\psi_2(x) + \psi_2^*(x)\psi_1(x)$ .<sup>5</sup> So the visibility  $\mathcal{V}$  can be just read off from Eq. (12) as  $\langle d_1|d_2\rangle$ . It is straightforward to check that if one starts from Eq. (4) instead of Eq. (11), one obtains the same visibility.

The quantons hitting the screen can be divided into two sub-ensembles according the ancilla states  $|a_1\rangle$  and  $|a_2\rangle$ . Quantons correlated with the ancilla state  $|a_1\rangle$  are in the state

$$\langle a_1 | \Psi_f \rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle|p_1\rangle + |\psi_2\rangle|p_2\rangle). \quad (13)$$

In this state the quanton path amplitudes are correlated with orthogonal states of the path-detector. Hence these quantons will not show any interference, as can be checked by evaluating  $|\langle a_1 | \langle x | \Psi_f \rangle|^2$ . However, for each of these quantons, measuring an observable of the path-detector whose non-degenerate eigenstates are  $|p_1\rangle, |p_2\rangle$ , unambiguously tells us which slit the quanton passed through.

Quantons correlated with the ancilla state  $|a_2\rangle$  are in the state

$$\langle a_2 | \Psi_f \rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)|q_1\rangle, \quad (14)$$

and the probability distribution of these quantons on the screen is given by

$$|\langle a_2 | \langle x | \Psi_f \rangle|^2 = \frac{1}{2}|\langle x | \psi_1 \rangle + \langle x | \psi_2 \rangle|^2. \quad (15)$$

These quantons will show full interference. All of this can also be verified in an experiment by correlating the quantons hitting the screen with the measurement results of the ancilla. There is a subtle point however, which should be mentioned here. Once the correlation between the quanton and the ancilla has been established by the state Eq. (11), it does not matter if the ancilla is measured first and the quanton is detected on the screen later or vice versa. Both cases will show exactly the same correlation in the measurement results. In fact, one may choose which operator of the ancilla to measure, well after the quanton has been registered on the screen. The state in Eq. (11) ensures that in an actual experiment, those correlations will be seen for sure. Such issues have also been discussed earlier.<sup>14,15</sup> In this sense, given the state in Eq. (11), one can talk about which states of the quantons are correlated with which states of the ancilla, without doing an actual measurement.

### C. Quantitative wave-particle duality

From Eq. (11), one can see that the fraction of quantons which contribute to interference, is  $\langle d_1 | d_2 \rangle$ . But the fraction of quantons giving rise to interference should intuitively be the *visibility* of the interference pattern. Indeed it is identical to the visibility obtained in the preceding subsection. So the interference visibility, denoted by  $\mathcal{V}$  is given by

$$\mathcal{V} = \langle d_1 | d_2 \rangle. \quad (16)$$

The above relation for visibility agrees with the one derived by Englert.<sup>9</sup>

We can define path-distinguishability  $\mathcal{D}_Q$  as the fraction of quantons for whom one can unambiguously tell which slit they passed through, *in the best case*. For a single quanton, path-distinguishability  $\mathcal{D}_Q$  is defined as the *maximum* probability with which one can unambiguously tell which slit it passed through. From Eq. (11), that

fraction is just  $1 - \langle d_1 | d_2 \rangle$ . Hence path-distinguishability is given by

$$\mathcal{D}_Q = 1 - \langle d_1 | d_2 \rangle, \quad (17)$$

and from Eq. (16) and (17) one can write

$$\mathcal{D}_Q + \mathcal{V} = 1. \quad (18)$$

The above result is a direct consequence of the fact that the fraction of quantons which give rise to interference is  $\langle d_1 | d_2 \rangle$ , and the fraction of quantons for which one can tell *for sure* which slit they came through, is  $1 - \langle d_1 | d_2 \rangle$ . It should be noted that these fractions can be inferred from Eq. (11) without doing an actual measurement on the ancilla. However, in order to find out for which quantons one can get full which-path information, an actual measurement on the ancilla must be performed. Getting a state  $|a_1\rangle$  of the ancilla would imply that for the particular quanton, one can tell for sure which slit it came from. If one gets a state  $|a_2\rangle$ , one cannot tell which slit that particular quanton came from.

If real experimental factors are taken into account, both visibility and path-distinguishability will have reduced values. Hence we can write the following inequality

$$\mathcal{D}_Q + \mathcal{V} \leq 1. \quad (19)$$

This is a duality relation which quantifies wave-particle duality, or complementarity, in a two-slit interference experiment.

In terms of the path-detector states  $|d_1\rangle$  and  $|d_2\rangle$ , the distinguishability introduced by Englert has the form<sup>9</sup>

$$\mathcal{D} = \sqrt{1 - |\langle d_1 | d_2 \rangle|^2}, \quad (20)$$

and Englert's distinguishability  $\mathcal{D}$  can be related to  $\mathcal{D}_Q$  by the relation

$$\mathcal{D}_Q = 1 - \sqrt{1 - \mathcal{D}^2} \quad (21)$$

Substituting the above in Eq. (19) gives

$$\mathcal{V} \leq \sqrt{1 - \mathcal{D}^2}. \quad (22)$$

Squaring both side, one get

$$\mathcal{V}^2 + \mathcal{D}^2 \leq 1, \quad (23)$$

which is identical to Eq. (1). So, when the quantity  $\mathcal{D}$  of Eq. (1) is evaluated for the case of pure detector states, the resulting relation between  $\langle d_1 | d_2 \rangle$  and  $\mathcal{V}$  is the same as the relation given by Eq. (19). In this sense Eqs. (1) and (19) are very closely analogous.

## IV. DISCUSSION

We have discussed a thought modification of a two-slit interference experiment, where the which-path information of quantons is extracted using UQSD. The analysis

shows that all the quantons passing through the double-slit, and registering on the screen, can be split into two sub-ensembles, depending on the measurement results of the ancilla. For quantons falling in the first sub-ensemble, one can unambiguously tell for each quanton which slit it passed through. These quantons do not contribute to interference.

For the quantons falling in the second sub-ensemble, one cannot tell which slit each of them passed through, but they all contribute to interference. Interference visibility intuitively should be just the fraction of quantons contributing fully to interference. For simply calculating

this fraction, without specifying which particular quantons can give full which-path information, one need not even do an actual measurement on the ancilla.

Using just the above arguments, we have derived a bound on the sum of the path-distinguishability and interference visibility. The derived duality relation expresses, in terms of unambiguous quantum state determination, a wave-particle trade-off closely analogous to a well-known duality relation.<sup>9</sup> UQSD can also be done for more than two states. Using UQSD, the duality relation has also been extended to interference experiments involving more than two slits.<sup>23,24</sup>

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